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Uniform-density cold neutron stars in general relativity

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Abstract. Nauenberg's uniform-density approximation method has been applied to study the stability of uniform-density neutron stars obeying (*a*) an equation of state due to Cohen, Langer, Rosen and Cameron, (*b*) an equation of state due to Bethe and Johnson and (*c*) extreme relativistic equations of state. It has been found that neutron stars may be stable in the first two cases but are unstable in the third case.

1. Introduction

Recently much interest has been evinced in the studies of the properties of neutron stars in the framework of general relativity. Oppenheimer and Volkoff (1939) initiated the studies by investigating the properties of a neutron core composed of an ideal (non-interacting) relativistic neutron gas. Zeldovich (1959, 1961), Cameron (1959), Sakyan (1963) and others have made contributions to the studies in this line by investigating different types of interactions possible in the dense neutron matter, as present in a neutron star. These studies are, however, based on numerical integration of the differential equations of hydrostatic equilibrium in general relativity obtained by Tolman (1934, 1939) and Oppenheimer and Volkoff (1939). Though this method is essential for the determination of the exact structure of the neutron star, it does not lead to a simple understanding of the properties of the star. Recently Nauenberg (1972) has developed an energy variational principle as an alternative approximate method and subsequently he, along with Chapline, (1973) has applied it to the uniform-density approximation for a cold star and obtained analytic expressions for the mass, the number of baryons and the radius of such a star. Applying these expressions to the equation of state of an ideal (non-interacting) relativistic neutron gas and also to the equation of state of a neutron gas where interneutronic interactions were taken into account, they have found that in both the cases the results obtained by the uniform-density approximation method agree fairly well with exact results obtained by numerical integration of the equations. The main advantage of the method is that the mathematics is fairly simple and a better insight into the problem may be obtained.

In this paper we apply the uniform-density approximation method first to two realistic equations of state of neutron gas and then extend the application of this method to the equations of state of extreme relativistic, completely degenerate, cold neutron gas. In § 2 of this paper we give a brief outline of the energy variational method in the uniform-density approximation. In § 3 we apply this method to the realistic equations of state for neutron gas suggested by (i) Cohen *et al* (1970) and (ii) Bethe and Johnson (1973). We also investigate the stability of the solutions against radical perturbations and find that in both cases certain stable solutions may be obtained. In § 4.1 we extend the application of the method to the equations of state for extreme relativistic, completely degenerate, cold ideal (non-interacting) neutron gas. In § 4.2, we study the effect of interneutronic interactions on the properties of the neutron matter by considering different types of interactions suggested by Zeldovich (1959, 1961) and Cameron (1959) and Sakyan (1963). In § 4.3 we investigate the stability of our solutions and find that in all the cases the solutions are unstable.

2. Energy variational method in the uniform-density approximation

In general relativity, the mass $M(r)$ and the number of baryons $A(r)$ of a spherically symmetric distribution of matter inside a sphere of coordinate radius r are given by

$$M(r) = 4\pi \int_0^r \rho(r) r^2 \, \mathrm{d}r \tag{1}$$

$$A(r) = 4\pi \int_0^r \frac{t(r) r^2 \, \mathrm{d}r}{[1-2GM(r)/c^2 r]^{1/2}} \tag{2}$$

where ρ is the mass density and t is the baryon number density. For constant-density spheres, Nauenberg carried out the integrations in (1) and (2) analytically and obtained the following expressions for the total mass M, total number of baryons A and the radius R:

$$M = \tfrac{4}{3}\pi \rho R^3 \tag{3}$$

$$A = 2\pi t \left(\frac{3c^2}{8\pi G\rho}\right)^{3/2} (\chi - \sin\chi\cos\chi) \tag{4}$$

$$R = \left(\frac{3c^2}{8\pi G\rho}\right)^{1/2} \sin\chi \tag{5}$$

where

$$\sin^2\chi = \frac{2GM}{Rc^2}. \tag{6}$$

The equilibrium condition is obtained by setting $\partial M/\partial\chi|_A = 0$ and is given by

$$\frac{p}{\rho c^2} = \zeta(\chi) \tag{7}$$

where pressure p is determined by

$$\frac{p}{c^2} = t\frac{\mathrm{d}\rho}{\mathrm{d}t} - \rho \tag{8}$$

and $\zeta(\chi)$ is a function of χ independent of the equation of state:

$$\zeta(\chi) = \frac{3\cos\chi}{\frac{9}{2}\cos\chi - \sin^3\chi/(\chi - \sin\chi\cos\chi)} - 1. \tag{9}$$

3. CLRC and BJ equations of state

We now apply the uniform-density approximation method to two realistic equations of state applicable to a neutron gas. Both the equations of state can be put in the form

$$p = \alpha(\nu - 1)t^{\nu} \tag{10}$$

$$\rho c^2 = m_n c^2 t + \alpha t^{\nu} \tag{11}$$

where m_n and t are the mass and particle density of the neutrons respectively and α and ν are two constants.

3.1. CLRC equation of state

To describe the behaviour of a dense neutron gas Cohen *et al* (1970) have suggested an equation of state of the form given by equations (10) and (11) where they have taken $\nu = 2{\cdot}9$. This equation holds for $\rho \geqslant 10^{14}\,\mathrm{g\,cm^{-3}}$. Taking $\alpha = 10^{-77}$ and using (10) and (11) in (3), (4) and (5) we get

$$M = \frac{c^4}{2G} L^{1/2} \sin^3\chi \tag{12}$$

$$A = 2\pi c^6 t L^{3/2}(\chi - \sin\chi\cos\chi) \tag{13}$$

$$R = c^2 L^{1/2} \sin\chi \tag{14}$$

where

$$L = \frac{3}{8\pi G(m_n c^2 t + 10^{-77} t^{2\cdot 9})}. \tag{15}$$

Variation of masses M and mA with density is studied by calculating M and mA for different particle densities starting from $t = 10^{38}$. The results are shown graphically in figure 1.

3.2. BJ equation of state

Bethe and Johnson (1973) also worked out an equation of state for cold, dense, neutron matter which is of the same form as given in equations (10) and (11). But in their equation which holds for $\rho > 3{\cdot}3 \times 10^{14}\,\mathrm{g\,cm^{-3}}$, they have taken $\nu = 2{\cdot}54$. Taking $\alpha = 10^{-64}$ and putting (10) and (11) in (3), (4) and (5) we get the expressions (12), (13) and (14) for M, A and R. But in this case

$$L = \frac{3}{8\pi G(m_n c^2 t + 10^{-64} t^{2\cdot 54})}. \tag{16}$$

In this case also we calculate M and mA for different particle densities starting from $t = 10^{38}$ and study the variation of these quantities with t. The results are shown graphically in figure 1.

3.3. Stability considerations

We now examine the stability of our equilibrium solution against radial perturbations.

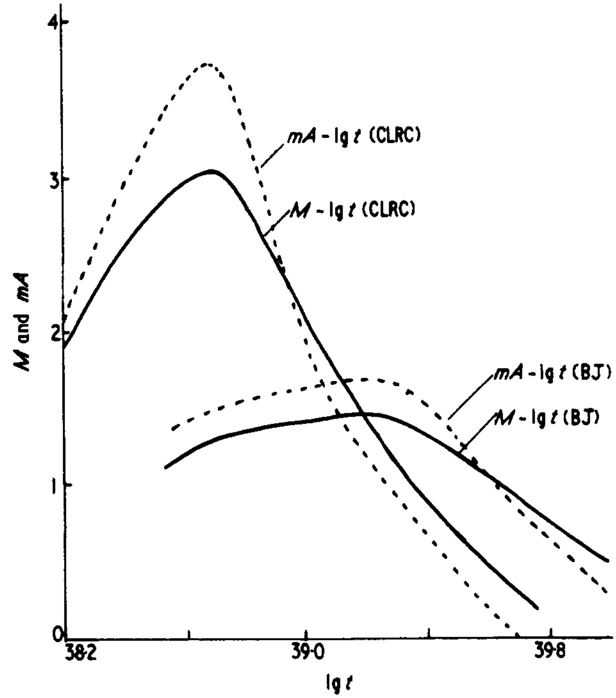

Figure 1. M–lg t and mA–lg t graphs for CLRC and BJ. M and mA are in units of solar masses.

The condition of stability in such a case is given by

$$\frac{\partial^2 M}{\partial \chi^2} > 0$$

ie

$$\tau > \tau_c(\chi) \tag{17}$$

where τ is the adiabatic index

$$\tau = \left(1 + \frac{\rho c^2}{p}\right)\frac{\mathrm{d}p}{c^2\,\mathrm{d}\rho} \tag{18}$$

and τ_c is a function of χ only.

$$\tau_c = (\zeta+1)\left[1 + \frac{(3\zeta+1)}{2}\left(\frac{(\zeta+1)}{6\zeta}\tan^2\chi - 1\right)\right]. \tag{19}$$

We calculate τ and τ_c for different values of $\zeta(\chi)$ for both the equations of state and plot τ–$\zeta(\chi)$ and τ_c–$\zeta(\chi)$ on the same graph (figure 2). From the figure it is seen that in the case of the CLRC equation of state the two curves intersect at $t = 4{\cdot}62 \times 10^{38}$, suggesting that condition (17) for stability is valid for $t \leqslant 4{\cdot}62 \times 10^{38}$. For the BJ equation of state the two curves intersect at $t = 1{\cdot}68 \times 10^{39}$ which indicates that the stability of our solution is possible only for $t \leqslant 1{\cdot}68 \times 10^{39}$. We have also calculated the fractional binding energy $(M - mA)/mA$ for different particle densities for both equations of state. Now from our results (figure 3) it is seen that for the CLRC equation

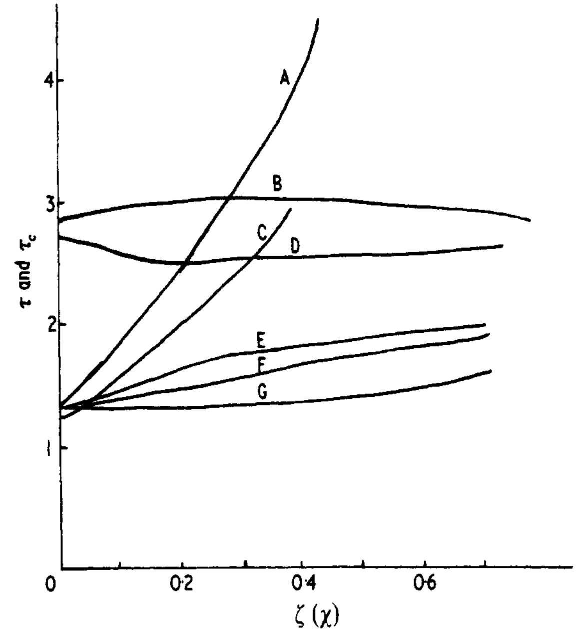

Figure 2. Graphs of $\zeta(\chi)$ against τ and τ_c. A, τ_c–$\zeta(\chi)$; B, τ–$\zeta(\chi)$ (CLRC); C, τ–$\zeta(\chi)$ (Cameron 1959 and Sakyan 1963); D, τ–$\zeta(\chi)$(BJ); E, τ–$\zeta(\chi)$ (Zeldovich 1961); F, τ–$\zeta(\chi)$ (Zeldovich 1959); G, τ–$\zeta(\chi)$ (without interaction).

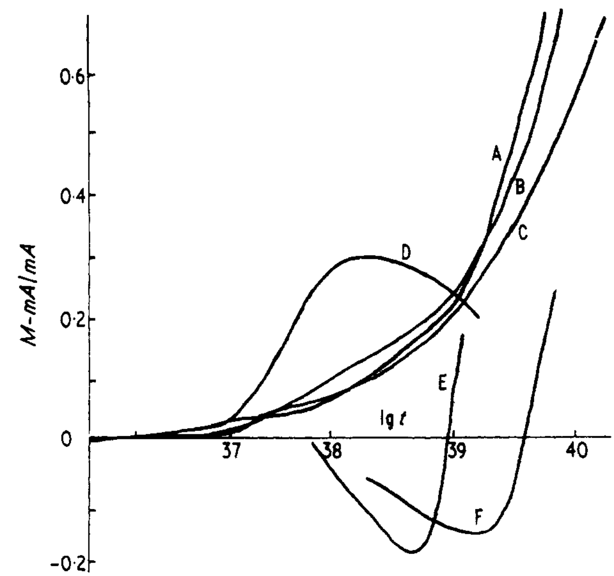

Figure 3. Graphs of $M-mA/mA$ against $\lg t$. A, Zeldovich (1961); B, Zeldovich (1959); C, without interaction; D, Cameron (1959) and Sakyan (1961); E, CLRC; F, BJ (M measured in solar masses).

of state the binding fraction is minimum for $t = 4{\cdot}57 \times 10^{38}$. On the other hand M–t and mA–t curves (figure 1) show prominent peaks at this particle density. Thus we may conclude that a uniform-density neutron star whose matter distribution obeys a

CLRC type of equation of state may have stable equilibrium configuration for $t \leqslant 4{\cdot}57 \times 10^{38}$. For the BJ equation of state the binding fraction is minimum for $t = 1{\cdot}64 \times 10^{39}$. At this point also both M–t and mA–t curves show prominent peaks (figure 1). Thus it may be possible for uniform-density stable neutron stars to exist, obeying a BJ type of equation of state for $t \leqslant 1{\cdot}64 \times 10^{39}$. For $t = 1{\cdot}64 \times 10^{39}$, $M_c = 1{\cdot}48 m_\odot$, $R_c = 6{\cdot}1$ km for the BJ type of equation of state and for $t = 4{\cdot}57 \times 10^{38}$, $M_c = 3{\cdot}06 m_\odot$, $R_c = 11{\cdot}3$ km for the CLRC type of equation of state.

4. Extreme relativistic neutron gas

4.1. *Without interaction*

We now extend the application of the uniform-density approximation method to the extreme relativistic neutron gas. The equation of state for an extreme relativistic, completely degenerate ideal (non-interacting) cold neutron gas having rest mass (the rest mass is considered following Inman (1965) since we are dealing with an extremely high density) is given by

$$p = \frac{1}{4}\left(\frac{6\pi^2}{g}\right)^{1/3} \hbar c (N/V)^{4/3} = K t^{4/3} \tag{20}$$

$$\rho c^2 = m_n c^2 t + 3K t^{4/3} \tag{21}$$

where

$$K = \frac{1}{4}\left(\frac{6\pi^2}{g}\right)^{1/3} \hbar c, \qquad t = N/V. \tag{22}$$

For neutrons $g = 1/2$, $m_n = 1{\cdot}67 \times 10^{-24}$ g and $K = 2{\cdot}45 \times 10^{-17}$ erg cm. Using (20) and (21) in (3), (4) and (5) one gets

$$M = \frac{c^4}{2G} H^{1/2} \sin^3\chi \tag{23}$$

$$A = 2\pi c^6 t H^{3/2} (\chi - \sin\chi \cos\chi) \tag{24}$$

$$R = c^2 H^{1/2} \sin\chi \tag{25}$$

where

$$H = \frac{3}{8\pi G (m_n c^2 t + 3K t^{4/3})}. \tag{26}$$

We study the properties of a neutron star whose matter obeys the equations (20) and (21), by calculating masses M and mA for different particle densities starting from $t = 10^{36}$. The results are shown graphically in figure 4.

4.2. *With interactions*

Since the density of neutron stars is extremely high, the nuclear forces acting between them cannot be ignored. Also corrections must be made for the presence of protons,

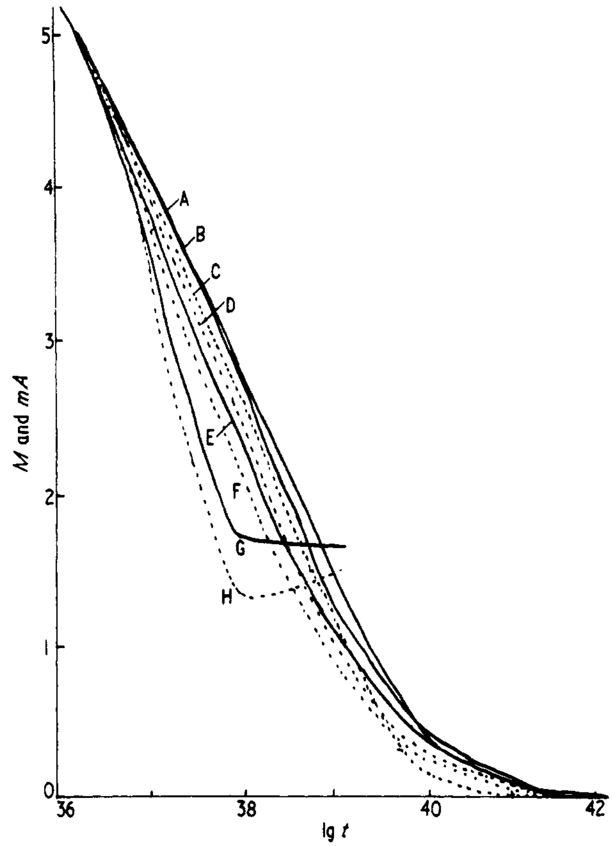

Figure 4. Graphs of M and mA against lg t. A, Zeldovich (1961); B, Zeldovich (1959); C, Zeldovich (1961); D, Zeldovich (1959); E, without interaction; F, without interaction; G, Cameron (1959) and Sakyan (1961); H, Cameron (1959) and Sakyan (1961). Full curves, M used, broken curves mA used. Both are in units of solar masses.

hyperons etc. Since the exact nature of the nuclear forces at short distances is, at present, not known, different workers have, from different considerations, put forward corrections for a neutron gas with interactions. Following Inman (1965) we will consider here three such corrections, as suggested by (i) Zeldovich (1959), (ii) Zeldovich (1961) and (iii) Cameron (1959) and Sakyan (1963).

(i) Taking into account Zeldovich's correction for interactions we have the following equation of state for extreme relativistic, completely degenerate, cold neutron gas:

$$p = Kt^{4/3} + \frac{128}{27\pi} K'\beta(3\pi^2)^{5/3}\left(\frac{\hbar}{m_n c}\right)^5 t^{5/3} \tag{27}$$

$$\rho c^2 = m_n c^2 t + 3Kt^{4/3} + \frac{64}{9\pi} K'\beta(3\pi^2)^{5/3}\left(\frac{\hbar}{m_n c}\right)^5 t^{5/3} \tag{28}$$

where

$$K = \frac{1}{4}\left(\frac{6\pi^2}{g}\right)^{1/3} \hbar c; \qquad K' = \frac{m_n^4 c^5}{32\pi^2 \hbar^3}; \qquad t = \frac{N}{V}. \tag{29}$$

Using (27) and (28) in (3), (4) and (5) we get,

$$M = \frac{c^4}{2G} H_1^{1/2} \sin^3\chi \tag{30}$$

$$A = 2\pi c^6 t H_1^{3/2}(\chi - \sin\chi\cos\chi) \tag{31}$$

$$R = c^2 H_1^{1/2} \sin\chi \tag{32}$$

where

$$H_1 = \frac{3}{8\pi G[m_n c^2 t + 3Kt^{4/3} + (64/9\pi)K'\beta(3\pi^2)^{5/3}(\hbar/m_n c)^5 t^{5/3}]}. \tag{33}$$

Taking $\beta = 1$ we now study the variation of M and mA with changing particle density by calculating M and mA for different values of t starting from $t = 10^{36}$. The results are shown graphically in figure 4. From figure 4 it is clear that the effect of introducing the interaction term is to have larger masses for the same particle densities.

(ii) A second type of correction suggested by Zeldovich (1961) for the above type of neutron matter leads to the following equation of state:

$$p = Kt^{4/3} + \frac{64}{9\pi} K'\beta(3\pi^2)^2 \left(\frac{\hbar}{m_n c}\right)^6 t^2 \tag{34}$$

$$\rho c^2 = m_n c^2 t + 3Kt^{4/3} + \frac{64}{9\pi} K'\beta(3\pi^2)^2 \left(\frac{\hbar}{m_n c}\right)^6 t^2 \tag{35}$$

where K, K' and t are given by equation (29).

Using (34) and (35) in (3), (4) and (5) we get expressions (30), (31) and (32) for M, A and R respectively. But in this case

$$H_1 = \frac{3}{8\pi G[m_n c^2 t + 3Kt^{4/3} + (64/9\pi)K'\beta(3\pi^2)^2(\hbar/m_n c)^6 t^2]}. \tag{36}$$

For this case we take $\beta = 3$ and calculate M and mA as before for different particle densities starting from $t = 10^{36}$. The results are shown in figure 4.

(iii) The third type of correction for dense neutron matter we consider is due to Cameron (1959) and Sakyan (1963). The equation of state in this case is of the form:

$$p = Kt^{4/3} + K'\left[39{\cdot}9(3\pi^2)^{8/3}\left(\frac{\hbar}{m_n c}\right)^8 t^{8/3} - 10{\cdot}1(3\pi^2)^2\left(\frac{\hbar}{m_n c}\right)^6 t^2\right] \tag{37}$$

$$\rho c^2 = m_n c^2 t + 3Kt^{4/3} + K'\left[23{\cdot}9(3\pi^2)^{8/3}\left(\frac{\hbar}{m_n c}\right)^8 t^{8/3} - 10{\cdot}1(3\pi^2)^2\left(\frac{\hbar}{m_n c}\right)^6 t^2\right] \tag{38}$$

where K, K' and t are given by equation (29).

Using (37) and (38) in (3), (4) and (5) we get expressions (30), (31) and (32) for M, A and R respectively. But in this case

$$H_1 = \frac{3}{8\pi G\{m_n c^2 t + 3Kt^{4/3} + K'[23{\cdot}9(3\pi^2)^{8/3}(\hbar/m_n c)^8 t^{8/3} - 10{\cdot}1(3\pi^2)^2(\hbar/m_n c)^6 t^2]\}}. \tag{39}$$

As before in this case also we calculate M and mA for different particle densities starting from $t = 10^{36}$ and study the variation of M and mA with t by plotting M–lg t and mA–lg t graphs (figure 4).

4.3. Stability considerations

We now investigate the stability of a neutron star composed of dense extreme relativistic neutron gas obeying any of the four equations of state we have considered above. As in § 3.3 in this case also we calculate τ and τ_c for different particle densities for all the four equations of state. τ–$\zeta(\chi)$ and τ_c–$\zeta(\chi)$ curves for all the four cases are shown in figure 2. From the figure it is clear that in all the four cases $\tau < \tau_c$ throughout the range of particle densities considered. This indicates that there cannot be any stable solution in this range. We have also calculated the binding fraction energy $(M - mA)/mA$ for different particle densities for all the four cases. The results are shown in figure 3. From the graph it is seen that none of the curves show a minimum of this quantity which also suggests that there cannot be any stable configuration within the range considered.

5. Conclusion

Applying uniform-density approximation method we have shown that it may be possible for stable neutron stars to exist obeying CLRC- and BJ-types of equations of state. We have also calculated maximum masses (M_c), mA and radius R_c for such stable stars. A comparison of our results with numerical solutions of the differential equation of hydrostatic equilibrium in general relativity indicates reasonable agreement. We have also extended the application of the uniform-density approximation method to neutron stars composed of extreme relativistic, completely degenerate, cold neutron gas and found that such a star is not stable.

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